

# HOMOTOPY TYPES OF SPACES OF SUBMANIFOLDS OF $\mathbb{R}^n$

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**ABSTRACT.** We compute the homotopy type of the space of proper  $d$ -dimensional submanifolds of  $\mathbb{R}^n$  with a smooth version of the Fell topology. Our methods allow us to compute the homotopy type of the space of submanifolds with summable labels too, and to give a new proof of the Galatius–Randal-Williams theorem on the homotopy type of their space of submanifolds.

## 1. INTRODUCTION

In [GMTW09], the classifying space of the  $d$ -dimensional cobordism category was found to be homotopy equivalent to a delooping of the infinite loop space associated to the Thom spectrum  $\mathbf{MTO}(d)$ , whose  $n$ -th space is the Thom space of the affine Grassmannian  $\gamma_{d,n}^\perp$  of  $d$ -planes in  $\mathbb{R}^n$ , seen as a vector bundle over the linear Grassmannian  $\mathrm{Gr}_d(\mathbb{R}^n)$ . This was proven again with different methods by Galatius and Randal-Williams [GRW10], who introduced the space  $\Psi_d(\mathbb{R}^n)$  of submanifolds of  $\mathbb{R}^n$ . A crucial step in their proof was proving that the inclusion

$$\mathrm{Th}(\gamma_{d,n}^\perp) \hookrightarrow \Psi_d(\mathbb{R}^n) \quad (1.1)$$

is a weak homotopy equivalence.

$\Psi_d(\mathbb{R}^n)$  is a topological space whose underlying set  $\psi_d(\mathbb{R}^n)$  is the subset of the power set of  $\mathbb{R}^n$  consisting of (possibly empty, possibly non-compact) proper  $d$ -submanifolds of  $\mathbb{R}^n$  without boundary. Here, a *proper subset of  $\mathbb{R}^n$*  is a subset of  $\mathbb{R}^n$  whose intersection with any compact subset is compact. A subset of  $\mathbb{R}^n$  is proper if and only if it is closed.

In this paper, we study a different topology of the set  $\psi_d(\mathbb{R}^n)$  and, via a counterpart to (1.1), show that  $\psi_d(\mathbb{R}^n)$  with this alternative topology is weakly contractible. We also study the set  $\psi_d(\mathbb{R}^n; X)$  of proper  $d$ -dimensional submanifolds of  $\mathbb{R}^n$  with summable labels on an abelian topological monoid  $X$ . We start by defining the new topology on  $\psi_d(\mathbb{R}^n)$ , the differential Fell topology, and explain how it relates to the topology on  $\Psi_d(\mathbb{R}^n)$ . The definition of the topology on  $\psi_d(\mathbb{R}^n; X)$  is postponed until Definition 2.3.

The set of closed subsets of a topological space  $X$  can be endowed with several topologies, of which the Fell topology [Fel62] is the most convenient for us. To describe it define, for each subset  $U \subset X$ ,

$$U^- = \{A \in \mathrm{CL}(X) \mid A \cap U \neq \emptyset\}, \quad U^+ = \{A \in \mathrm{CL}(X) \mid A \subset U\}.$$

Then the Fell topology has as subbasis the collection of all subsets  $U^-$  with  $U$  an open subset of  $X$  and  $U^+$  with  $U$  the complement of a compact subset of  $X$ . The reader more familiar with topologies in function spaces may gain some intuition by

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knowing that if  $X$  is metrizable, then there exists a metric  $d$  on  $X$  for which the function

$$\mathrm{CL}(X) \longrightarrow \mathrm{map}(X, \mathbb{R}) \quad (1.2)$$

given by  $A \mapsto d(A, -)$  is an embedding if the right hand-side is endowed with the compact-open topology [Bee88, Theorem 2(d)].

The subset  $\psi_d(\mathbb{R}^n) \subset \mathrm{CL}(\mathbb{R}^n)$  can be endowed with the subspace topology, but this of course does not take into account the smooth structure of the submanifolds. This is addressed by considering  $\psi_d(\mathbb{R}^n)$  not as a subset of  $\mathrm{CL}(\mathbb{R}^n)$ , but as a subset

$$\psi_d(\mathbb{R}^n) \subset \mathrm{CL}(\mathbb{R}^n \times \mathrm{Gr}_d(\mathbb{R}^n)),$$

using the Gauss map that sends a submanifold  $W$  to  $\{(x, T_x W) \mid x \in W\}$ .

We denote by  $\tilde{\Psi}_d(\mathbb{R}^n)$  the set  $\psi_d(\mathbb{R}^n)$  endowed with the subspace topology of the latter inclusion, and we refer to it as the *space of merging submanifolds* of  $\mathbb{R}^n$ .

This topology is still coarser than the topology in  $\Psi_d(\mathbb{R}^n)$ , but it is very close to it [Can15]: A sequence of compact submanifolds in  $\tilde{\Psi}_d(\mathbb{R}^n)$  converging to a compact submanifold  $W$  eventually takes values in covering spaces of  $W$ . If we refine the topology  $\tilde{\Psi}_d(\mathbb{R}^n)$  imposing these covering spaces to be single-sheeted (i.e., diffeomorphisms), then we arrive to the topology  $\Psi_d(\mathbb{R}^n)$  defined by Galatius and Randal-Williams.

The following is the main theorem of this paper. Let  $\mathcal{L}_d(\mathbb{R}^n)$  and  $\tilde{\mathcal{L}}_d(\mathbb{R}^n)$  be the subspaces of  $\Psi_d(\mathbb{R}^n)$  and  $\tilde{\Psi}_d(\mathbb{R}^n)$ , consisting only on (possibly empty) unions of parallel affine planes together with the empty set. Let  $\tilde{\mathcal{L}}_d(\mathbb{R}^n; X)$  be the subspace of  $\tilde{\Psi}_d(\mathbb{R}^n; X)$  of (possibly empty) unions of parallel affine planes and locally constant labels (in the notation of Definition 2.3,  $\alpha$  is locally constant).

**Theorem.** *The inclusions*

$$\begin{aligned} \mathcal{L}_d(\mathbb{R}^n) &\hookrightarrow \Psi_d(\mathbb{R}^n) \\ \tilde{\mathcal{L}}_d(\mathbb{R}^n) &\hookrightarrow \tilde{\Psi}_d(\mathbb{R}^n) \\ \tilde{\mathcal{L}}_d(\mathbb{R}^n; X) &\hookrightarrow \tilde{\Psi}_d(\mathbb{R}^n; X) \end{aligned}$$

*are part of a strong deformation retraction. In addition, the inclusion  $\mathrm{Th}(\gamma_d^\perp(\mathbb{R}^n)) \hookrightarrow \mathcal{L}_d(\mathbb{R}^n)$  is part of a strong deformation retraction and  $\tilde{\mathcal{L}}_d(\mathbb{R}^n)$  is weakly contractible.*

Though the main contribution of this theorem is the computation of the homotopy type of  $\tilde{\Psi}_d(\mathbb{R}^n)$  and  $\tilde{\Psi}_d(\mathbb{R}^n; X)$ , the part referring to  $\Psi_d(\mathbb{R}^n)$  has its own interest: Galatius and Randal-Williams showed that (1.1) is a weak homotopy equivalence, and a formal corollary of our main theorem improves their result as follows:

**Corollary.** *The inclusion (1.1) is a strong deformation retraction, so in particular  $\Psi_d(\mathbb{R}^n)$  has the homotopy type of a CW-complex.*

The assignments  $\Psi_d(-)$ ,  $\tilde{\Psi}_d(-)$  and  $\tilde{\Psi}_d(-; X)$  define sheaves on the site of manifolds and open embeddings (see Section 3.1 and [GRW10]). In [RW11], Randal-Williams proved the remarkable property that  $\Psi_d(-)$  is a microflexible sheaf. In Section 7 we show that  $\tilde{\Psi}_d(-)$  and  $\tilde{\Psi}_d(-; X)$  are not microflexible, so the results of [RW11, §3-6] on  $\Psi_d(-)$  (which are based on [Gal11]) do not generalize to  $\tilde{\Psi}_d(-)$  and  $\tilde{\Psi}_d(-; X)$ .

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## 2. SPACES OF SUBMANIFOLDS AND AN OVERVIEW OF THE PROOF

In this section we give a precise definition of the spaces  $\Psi_d(U)$  and  $\tilde{\Psi}_d(U)$  ([GRW10, §2], [Can15]) and we give a some comments on the proof. For the sake of clarity, we will give a  $C^1$ -version of this topology as in [BM11]. We start setting up some conventions:

We denote by  $d_0$  the Euclidean distance in  $\mathbb{R}^n$ , by  $d_1$  the distance on  $\text{Gr}_d(\mathbb{R}^n)$  given by

$$d_1(L, L') = \max_{v \in S(L)} \min_{w \in S(L')} \{\text{angle}(v, w)\} = \max_{w \in S(L')} \min_{v \in S(L)} \{\text{angle}(v, w)\},$$

where  $S(L)$  is the unit sphere in  $L$ , and by  $d = d_0 + d_1$  the distance in  $\mathbb{R}^n \times \text{Gr}_d(\mathbb{R}^n)$ . If  $f: L \rightarrow L'$  is a linear operator, then

$$\|f\| = \max_{v \in S(L)} \{\|f(v)\|\}.$$

If  $W \in \psi(U)$ , we denote by  $p: NW \rightarrow W$  the projection, which is covered by two bundle maps: its differential  $Dp: T(NW) \rightarrow TW$  and the canonical bundle isomorphism

$$\begin{array}{ccc} T(NW) & \xrightarrow{\alpha} & TW \oplus NW \\ \downarrow & & \downarrow \\ NW & \xrightarrow{p} & W. \end{array}$$

The differential  $Dp$  is the composition of  $\alpha$  with the projection onto  $TW$ , and we write  $\tau$  for the composition of  $\alpha$  with the projection onto  $NW$ .

Recall that there is a function  $\epsilon: W \rightarrow (0, \infty)$  such that the restriction

$$\exp_W^\epsilon: N^\epsilon W \rightarrow U$$

of the exponential map  $\exp_W: NW \rightarrow U$  to the subspace  $N^\epsilon W$  of vectors  $v$  of length at most  $\epsilon(p(v))$  is an embedding. In addition, the restriction of this map to each fibre

$$\exp_{W,x}^\epsilon: N_x^\epsilon W \rightarrow U$$

is a radial isometry. We denote by  $z: W \rightarrow NW$  the zero section, by  $T^\epsilon$  the image of  $N^\epsilon W$ , and by  $\pi$  the composition of  $(\exp_W^\epsilon)^{-1}$  and  $p$ . Summarizing:

$$\begin{array}{ccccc} NW & \supset & N^\epsilon W & & \\ p \downarrow & \nearrow z & \exp_W^\epsilon \downarrow \cong & & \\ W & \xleftarrow{\pi} & T^\epsilon & \subset & U \end{array}$$

**Definition 2.1.** The space  $\Psi_d(U)$  has as underlying set the collection of all proper  $d$ -dimensional submanifolds of  $U$ , together with the empty submanifold. Its topology is given by the following neighbourhood basis of any proper submanifold  $W$ : every compact subset  $K \subset U$  and every  $\epsilon > 0$  define a basic neighbourhood  $(K, \epsilon)^\Psi$  of  $W$ ; a submanifold  $W'$  belongs to  $(K, \epsilon)^\Psi$  if there is a section  $f$  of the normal bundle  $NW \rightarrow W$  such that

- (1)  $\exp_W(f(W)) \cap K = W' \cap K$  and
- (2)  $\|f(x)\| + \|\tau \circ D(f)(x)\| < \epsilon$  for all  $x \in W$  such that  $\exp_W \circ f(x) \in W' \cap K$ .

We now define the topology in  $\tilde{\Psi}_d(\mathbb{R}^n)$ , the only difference being that instead of requiring  $W' \cap K$  to be the image of a global section of  $NW$ , we only ask it to be the union of images of local sections of  $NW$  whose domains cover  $W$ . The fact that this is the topology in the space of merging submanifolds (p. 2) is shown in [Can15, Theorem 1].

**Definition 2.2.** The space  $\tilde{\Psi}_d(U)$  has the same underlying set as  $\Psi_d(U)$ , with neighbourhood basis of a proper submanifold  $W$ : every compact subset  $K \subset U$  and every  $\epsilon > 0$  define a basic neighbourhood  $(K, \epsilon)^{\tilde{\Psi}}$  of  $W$ ; a submanifold  $W'$  belongs to  $(K, \epsilon)^{\tilde{\Psi}}$  if there is a subset  $Q \subset NW$  such that the composite  $Q \subset NW \rightarrow W$  is a smooth covering map onto  $W \cap K$  and

- (1)  $\exp_W(Q) \cap K = W' \cap K$ ,
- (2)  $\|f(x)\| + \|\tau \circ D(f)(x)\| < \epsilon$  for each local section  $f$  of the covering map  $q$ .

Let  $U \subset \mathbb{R}^n$  be an open subset, let  $X$  be an abelian topological monoid with unit and let  $\psi_d(U; X)$  be the set of pairs  $(W, \alpha)$  with  $W \in \psi_d(U)$  and  $\alpha: W \rightarrow X$  a continuous map.

**Definition 2.3.** The space  $\tilde{\Psi}_d(U; X)$  has underlying set  $\psi_d(U; X)$ , with the following neighbourhood basis of a pair  $(W, \alpha)$ : every compact subset  $K \subset U$ , every  $\epsilon > 0$  and every open neighbourhood  $A$  of  $\alpha|_{W \cap K}$  in  $\text{map}(W \cap K, X)$  define a basic neighbourhood  $(K, \epsilon, A)^{\Psi(-, X)}$  of  $W$ ; a submanifold  $W'$  belongs to  $(K, \epsilon, A)^{\Psi(-, X)}$  if there is a subset  $Q \subset NW$  such that the composite  $Q \subset NW \rightarrow W$  is a smooth covering map onto  $W \cap K$  and

- (1)  $\exp_W(Q) \cap K = W' \cap K$ ,
- (2)  $\|f(x)\| + \|\tau \circ D(f)(x)\| < \epsilon$  for each local section  $f$  of the covering map  $q$ .
- (3) The map  $\beta: W \cap K \rightarrow X$ , given by  $\beta(x) = \sum_{y \in q^{-1}(x)} \alpha'(y)$ , belongs to  $A$ .

**Remark 2.4.** (1) If  $W' \in (K, \epsilon)^{\Psi}$  (respectively,  $W' \in (K, \epsilon)^{\tilde{\Psi}}$ ) and  $f$  is a global (resp. local) section of the corresponding  $q: Q \rightarrow W$ , then [Can15, Lemma 3.1]

$$d_0(x, f(x)) + d_1(T_x W, T_{f(x)} W') < \epsilon \quad (2.1)$$

for all  $x \in W$  such that  $\exp_W \circ f(x) \in W' \cap K$ .

- (2) Condition (2) in both definitions says that  $f$  is  $\epsilon$ -close to the zero section in the  $C^1$ -topology. One can instead impose that condition in the  $C^\infty$ -topology. This would give, in the case of  $\Psi_d(U)$ , the actual definition given by Galatius and Randal-Williams. This change does not change the weak homotopy type of  $\Psi_d(\mathbb{R}^n)$  and  $\tilde{\Psi}_d(\mathbb{R}^n)$ .
- (3) When  $d = 0$ , the subspace of  $\Psi_0(\mathbb{R}^n)$  consisting of 0-submanifolds contained in the unit ball, is the unordered configuration space on the unit ball, whereas that subspace in  $\tilde{\Psi}_0(\mathbb{R}^n)$  is the Ran space of the unit ball. Think now of  $S^n$  as the one-point compactification of  $\mathbb{R}^n$ , with the north pole as the point at infinity. The space  $\tilde{\Psi}_0(\mathbb{R}^n)$  is the subspace of the Ran space of  $S^n$  consisting on configurations that contain the north pole. The space  $\tilde{\Psi}_0(\mathbb{R}^n; \mathbb{N} \setminus \{0\})$  is the infinite symmetric product of  $S^n$  with the north pole as the basepoint.
- (4) There is a natural identification of  $\Psi(U)$  with the subspace of  $\tilde{\Psi}_d(U; \mathbb{Z})$  of pairs of the form  $(W, 1)$ , and a natural identification of  $\tilde{\Psi}_d(U)$  with the subspace of  $\tilde{\Psi}_d(U; \mathbb{Z})$  of pairs of the form  $(W, 0)$ .

We will give the proof of the first part of the main theorem only for the inclusion  $\tilde{\mathcal{L}}_d(\mathbb{R}^n) \hookrightarrow \tilde{\Psi}_d(\mathbb{R}^n)$ , as the proof for the other two inclusions is exactly the same. Here is an overview of the proof: Let  $\tilde{\Psi}_d(\mathbb{R}^n)^\delta \subset \tilde{\Psi}_d(\mathbb{R}^n)$  be the subspace of those submanifolds  $W$  for which the diameter of the image of the Gauss map

$$\text{Gauss}: W \rightarrow \text{Gr}_d(\mathbb{R}^n)$$

is at most  $\delta$ . Then, in Proposition 4.8 we will prove that when  $\delta > 0$ , the inclusion  $\tilde{\Psi}_d(\mathbb{R}^n)^\delta \subset \tilde{\Psi}_d(\mathbb{R}^n)$  is a homotopy equivalence that fixes  $\tilde{\mathcal{L}}_d(\mathbb{R}^n) = \tilde{\Psi}_d(\mathbb{R}^n)^0 \subset$

$\tilde{\Psi}_d(\mathbb{R}^n)^\delta$ . Then, in Proposition 5.4 we will assume that  $\delta$  is small, and we will continuously assign to each submanifold  $W \in \tilde{\Psi}_d(\mathbb{R}^n)^\delta$ , a  $d$ -plane  $\mu(W)$  such that  $\mu(W)^\perp$  is transverse to  $W$ . In Proposition 5.5 we will construct a deformation retraction  $\tilde{\Psi}_d(\mathbb{R}^n)^\delta \rightarrow \tilde{\mathcal{L}}_d(\mathbb{R}^n)$  that stretches out  $W$  in the direction  $\mu(W)$ . Finally, in Propositions 6.1 and 6.2 we find the homotopy type of  $\mathcal{L}(\mathbb{R}^n)$  and  $\tilde{\mathcal{L}}_d(\mathbb{R}^n)$ .

**Remark 2.5.** The first part of the proof (Section 4) can be significantly shortened if one is only interested in proving that the inclusion  $\tilde{\mathcal{L}}_d(\mathbb{R}^n) \subset \tilde{\Psi}_d(B^n)$  is a weak homotopy equivalence (see Remark 4.2).

### 3. A TECHNICAL LEMMA AND AN ASSUMPTION

**3.1. The action of a space of embeddings on the space of merging submanifolds.** The next lemma can be proven as in [GRW10, §2.2], making the appropriate modifications. Instead, we will take advantage of knowing that the topology of  $\tilde{\Psi}_d(U)$  is induced by the Fell topology.

**Lemma 3.1.** *Let  $U, V$  be open subsets of  $\mathbb{R}^n$  and let  $\text{Emb}(U, V)$  be the space of embeddings of  $U$  into  $V$  with the  $C^1$  compact-open topology. Then the map*

$$\text{Emb}(U, V) \times \tilde{\Psi}_d(V) \longrightarrow \tilde{\Psi}_d(U) \quad (3.1)$$

*given by sending a pair  $(f, W)$  to  $f^{-1}(W)$  is continuous.*

*Proof.* Let first  $X, Y$  be metric spaces. The function

$$\text{map}(X, \mathbb{R}) \rightarrow \text{CL}(X)$$

given by  $f \mapsto f^{-1}(0)$  is easily seen to be continuous if  $X$  is locally path-connected and in that case, it is a retraction of the map (1.2), hence a quotient map. Therefore, in the commutative diagram

$$\begin{array}{ccc} \text{Emb}(X, Y) \times \text{map}(Y, \mathbb{R}) & \longrightarrow & \text{map}(X, \mathbb{R}) \\ \downarrow & & \downarrow \\ \text{Emb}(X, Y) \times \text{CL}(Y) & \longrightarrow & \text{CL}(X) \end{array}$$

where the upper horizontal arrow is composition and the lower horizontal arrow is evaluation, the latter is continuous by the universal property of quotient maps.

Now, let  $U, V$  be open subsets of  $\mathbb{R}^n$ . Then the map in the statement fits in the commutative diagram

$$\begin{array}{ccc} \text{Emb}(U, V) \times \tilde{\Psi}_d(\mathbb{R}^n) & \longrightarrow & \tilde{\Psi}_d(\mathbb{R}^n) \\ \downarrow & & \downarrow \\ \text{Emb}(U \times \text{Gr}_d(\mathbb{R}^n), V \times \text{Gr}_d(\mathbb{R}^n)) \times \text{CL}(V \times \text{Gr}_d(\mathbb{R}^n)) & \longrightarrow & \text{CL}(U \times \text{Gr}_d(\mathbb{R}^n)). \end{array}$$

The lower horizontal map has been shown to be continuous, and the left vertical map is easily seen to be continuous too. Therefore the upper horizontal map is continuous because the right vertical map is an embedding.  $\square$

**Lemma 3.2.** *Let  $U, V$  be open subsets of  $\mathbb{R}^n$  and let  $\text{Emb}(U, V)$  be the space of embeddings of  $U$  into  $V$  with the  $C^1$  compact-open topology. Then the map*

$$\text{Emb}(U, V) \times \tilde{\Psi}_d(V; X) \longrightarrow \tilde{\Psi}_d(U; X) \quad (3.2)$$

*given by sending a pair  $(f, W)$  to  $(f^{-1}(W), \alpha \circ f)$  is continuous.*

*Proof.* Let  $(f, (W, \alpha)) \in \text{Emb}(U, V) \times \tilde{\Psi}_d(V; X)$  and let  $(K, \epsilon, A)$  be a neighbourhood of  $(f^{-1}(W), \alpha \circ f)$ . Let  $K' \subset V$  be a compact subset containing a relatively compact open neighbourhood  $B$  of  $f(K)$ . Let  $(K, B)$  be the open neighbourhood of  $f$  consisting of embeddings that send  $K$  into  $B$ . Let  $(K, \epsilon)$  be the open neighbourhood of  $f$  consisting of embeddings for which

$$\begin{aligned} \|f(x) - g(x)\| &< \epsilon && \text{for all } x \in K \\ \|Df(x) - Dg(x)\| &< \epsilon && \text{for all } x \in K. \end{aligned}$$

Then there are  $\delta_0, \delta_1 > 0$  and an  $A$  for which the image of  $((K, B) \cap (K, \delta_0)) \times (K', \delta_1, A)$  is contained in  $(K, \epsilon, A)$ .  $\square$

These maps preserve the subspaces  $\Psi_d(U)$  and  $\tilde{\Psi}_d(U)$  (see Remark 2.4 (4)). In particular, Lemma 3.2 gives another proof of Lemma 3.1 and of Section 2.2 in [GRW10].

**3.2. A more convenient assumption.** Let  $B^n$  be the open unit ball in  $\mathbb{R}^n$ . By the previous lemma, any diffeomorphism  $f: B^n \rightarrow \mathbb{R}^n$  induces a homeomorphism  $\tilde{\Psi}_d(B^n) \rightarrow \tilde{\Psi}_d(\mathbb{R}^n)$ . Our theorem will be proven by showing that the inclusion  $\tilde{\mathcal{L}}_d(B^n) \rightarrow \tilde{\Psi}_d(B^n)$  is a homotopy equivalence, where  $\tilde{\mathcal{L}}_d(B^n)$  consists on intersections of affine planes with the unit ball. The reason for this change is that it will be convenient to make use of diffeomorphisms between  $B^n$  and balls of different radius fixing the subspaces  $\tilde{\mathcal{L}}_d(B^n)$ , and this is more complicated we take  $\mathbb{R}^n$  instead.

#### 4. THE SPACE OF ALMOST LINEAR SUBMANIFOLDS

Recall that, for a submanifold  $W$  of  $B^n$ , the Gauss map

$$\text{Gauss}: W \rightarrow \text{Gr}_d(\mathbb{R}^n)$$

sends a point to its tangent plane. Let us write  $\text{diam}$  for diameter.

**Definition 4.1.** The space of  $\delta$ -almost linear submanifolds of  $B^n$  is the subspace  $\tilde{\Psi}_d(\mathbb{R}^n)^\delta \subset \tilde{\Psi}_d(B^n)$  of those submanifolds  $W$  such that  $\text{diam} \circ \text{Gauss}(W) < \delta$ .

In this section we prove that the inclusion  $\tilde{\Psi}_d(B^n)^\delta \subset \tilde{\Psi}_d(B^n)$  is a homotopy equivalence that fixes  $\tilde{\mathcal{L}}_d(\mathbb{R}^n)$  when  $\delta > 0$ .

**Remark 4.2.** To prove that this inclusion is a weak homotopy equivalence is easier: Let us see first that it is surjective on components: if  $W \in \tilde{\Psi}_d(B^n)$ , then there exists an  $\epsilon > 0$  such that for all  $0 < a \leq \epsilon$ ,  $(\frac{1}{a} \cdot W) \cap B^n$  is in  $\tilde{\Psi}_d(B^n)^\delta$ . Then, by (3.1), the isotopy  $t \mapsto (x \mapsto \frac{1}{1+(\epsilon-1)t})$  induces a path from  $(\frac{1}{\epsilon} \cdot W) \cap B^n$  to  $W$ .

Now, given a lifting problem

$$\begin{array}{ccc} \partial D^n & \xrightarrow{f} & \tilde{\Psi}_d(B^n)^\delta \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{g} & \tilde{\Psi}_d(B^n) \end{array}$$

let

$$\epsilon(x) = \sup\{\epsilon \in (0, 1] \mid \forall a < \epsilon, \frac{1}{a}f(x) \in \tilde{\Psi}_d(B^n)^\delta\}.$$

One can use that  $D^n$  is compact to find an  $\epsilon$  as above that works for all the values of  $g$  at once. Then, using this  $\epsilon$ , one replaces this lifting problem by a homotopy equivalent lifting problem in which the lift exists on the nose.

**4.1. Another technical lemma.** Let  $F_{\geq}([0, 1))$  be the set of bounded non-decreasing real functions from  $[0, 1)$  to  $[0, \infty)$  that preserve 0, and let  $C_{>}([0, 1))$  be the subset of continuous increasing functions. We topologize these spaces as follows: For each positive real number  $\epsilon$ , there is a basic neighbourhood  $(\epsilon)^F$  of a function  $f$ . Another function  $g$  is in the  $(\epsilon)$ -neighbourhood of  $f$  if for all  $s \in [\epsilon, 1 - \epsilon]$ ,  $f(s - \epsilon) - \epsilon \leq g(s) \leq f(s + \epsilon) + \epsilon$ . The subspace  $C_{>}([0, 1))$  has the compact-open topology.

**Definition 4.3.** For each  $f \in F_{\geq}([0, 1))$ , define  $\rho(f): [0, 1) \rightarrow [0, \infty)$  as

$$\rho(f)(x) = \int_x^{\sqrt{x}} f(y) dy.$$

**Lemma 4.4.** (1) If  $f \in F_{\geq}([0, 1))$ , then  $\rho(f) \in C_{\geq}([0, 1))$  and  $f \leq \rho(f)$ ;  
 (2) if  $f \in F_{>}([0, 1))$ , then  $\rho(f) \in C_{>}([0, 1))$  and  $f < \rho(f)$ ;  
 (3) if  $f = 0$ , then  $\rho(f) = 0$ .

*Proof.* The third point is obvious. For the other two, observe first that the value of  $\rho(f)$  on  $x$  is the mean value of  $f$  in the interval  $[x, \sqrt{x}]$ .

- (1) As  $f$  is non-decreasing, the mean value of  $f$  in  $[x, \sqrt{x}]$  is non-decreasing too, and it is bigger or equal than the value of  $f$  at the initial point of the interval.
- (2) If  $f$  is strictly increasing, then the mean value of  $f$  in  $[x, \sqrt{x}]$  is strictly increasing too, and strictly bigger than the value of  $f$  at the initial point of the interval.  $\square$

**Lemma 4.5.** The operator  $\rho: F_{\geq}([0, 1)) \rightarrow C_{\geq}([0, 1))$  is continuous.

*Proof.* If  $g$  is in the  $\epsilon$ -neighbourhood of  $f$ , then

$$\begin{aligned} & \left| \int_x^{\sqrt{x}} f(y) dy - \int_x^{\sqrt{x}} g(y) dy \right| \\ & \leq \int_0^1 |f(y) - g(y)| dy \\ & \leq \int_0^\epsilon |f(y) - g(y)| dy + \int_\epsilon^{1-\epsilon} |f(y) - g(y)| dy + \int_{1-\epsilon}^1 |f(y) - g(y)| dy \\ & \leq \epsilon + \int_\epsilon^{1-\epsilon} 2\epsilon dy + \int_{2\epsilon}^1 f(y) dy - \int_0^{1-2\epsilon} f(y) dy + \epsilon \\ & \leq 2\epsilon + 2\epsilon(1 - 2\epsilon) + \left( \int_{2\epsilon}^{1-\epsilon} f(y) dy + \int_{1-\epsilon}^1 f(y) dy \right) - \left( \int_0^\epsilon f(y) dy + \int_\epsilon^{1-2\epsilon} f(y) dy \right) + \epsilon \\ & \leq \epsilon + 2\epsilon(1 - 2\epsilon) + \left( \int_{2\epsilon}^{1-\epsilon} f(y) dy - \int_\epsilon^{1-2\epsilon} f(y) dy \right) + \left( \int_{1-\epsilon}^1 f(y) dy - \int_0^\epsilon f(y) dy \right) + \epsilon \\ & \leq 2\epsilon + 2\epsilon(1 - 2\epsilon) + (\epsilon) + (\epsilon) + \epsilon. \end{aligned} \quad \square$$

If we let  $\overline{X}$  denote the one-point compactification of a locally compact space  $X$ , then there is a continuous map

$$\text{OEmb}(X, Y) \longrightarrow \text{map}(\overline{Y}, \overline{X}) \quad (4.1)$$

from the space of open embeddings of  $X$  into  $Y$  to the mapping space between  $\overline{Y}$  and  $\overline{X}$ , both endowed with the compact-open topology (this is an adaptation of the second part of the proof of Theorem 4 in [Are46], cf. [Can14]). It is given by sending an embedding  $e$  to the map that sends a point  $y$  to  $e^{-1}(y)$  if the latter exists and to  $\infty$  otherwise.

The following lemma and its use was suggested to me by Abdó Roig.

**Lemma 4.6.** *For each  $\delta > 0$ , there exists a continuous function  $\sigma: F_{\geqslant}([0, 1]) \rightarrow (0, \frac{1}{2})$  such that  $f(\sigma(f)) < \delta$  for all  $f$  and  $\sigma(0) = 1$ .*

*Proof.* It is easy to see that the assignment  $f \mapsto (y \mapsto f(y) + \delta y)$  defines a continuous function

$$\tau: F_{\geqslant}([0, 1]) \mapsto F_{>}([0, 1])$$

with the property that  $\tau(f) \geqslant f$ . By Lemma 4.5, composing it with  $\rho$ , we obtain a continuous function to  $C_{>}([0, 1])$ .

Functions in  $C_{>}([0, 1])$  are not invertible, but they are open embeddings, hence the map (4.1) particularises in this case to a map

$$\text{inv}: C_{>}([0, 1]) \xrightarrow{f \mapsto f^{-1}} C_{\geqslant}([0, \infty]).$$

Therefore, if we write  $\text{ev}_{\delta}$  for the evaluation at  $\delta$ , then the composite

$$\sigma: F_{\geqslant}([0, 1]) \xrightarrow{\tau} F_{>}([0, 1]) \xrightarrow{\rho} C_{>}([0, 1]) \xrightarrow{\text{inv}} C_{\geqslant}([0, \infty]) \xrightarrow{\text{ev}_{\delta}} [0, 1]$$

is continuous. Writing  $g = \tau(f)$  and using that  $f < g$  and that  $g < \rho(g)$  (see Lemma 4.4),

$$f(\sigma(f)) = f(\rho(g)^{-1}(\delta)) < g(\rho(g)^{-1}(\delta)) < \rho(g)(\rho(g)^{-1}(\delta)) = \delta.$$

In addition, we have that if  $f = 0$ , then  $\sigma(f) = \rho(\tau(0))^{-1}(\delta) = \rho(\delta x)^{-1}(\delta) = 1$ .  $\square$

**4.2. The space of  $\delta$ -linear submanifolds.** Let us define

$$\vartheta: \tilde{\Psi}_d(B^n) \longrightarrow F_{\geqslant}([0, 1])$$

as the adjoint of the map  $\tilde{\Psi}_d(B^n) \times [0, 1] \rightarrow [0, \infty)$  that sends a pair  $(W, \epsilon)$  to  $\text{diam} \circ \text{Gauss}(W_{\epsilon})$ , where  $W_{\epsilon} = W \cap B(\epsilon)$  and  $B(\epsilon)$  is the ball of radius  $\epsilon$  centered at the origin.

**Lemma 4.7.** *The map  $\vartheta$  is continuous.*

*Proof.* Let  $(\epsilon)^F$  be a neighbourhood of  $\vartheta(W)$ . Let  $K(r) \subset B^n$  be the closed disc with radius  $r = 1 - \epsilon$ , and let  $(K(r), \epsilon)$  be a neighbourhood of  $W$ . If  $s < r$  and  $W' \in (D(s), \epsilon)^{\tilde{\Psi}}$ , then, by condition (1) and by (2.1) in the definition of the topology of  $\tilde{\Psi}_d(\mathbb{R}^n)$  the followig holds: for each  $y \in W'_s$ , there is an  $x \in W$  at distance at most  $\epsilon$  (so  $x \in W_{s+\epsilon}$ ), and such that  $T_y W'$  and  $T_x W$  are also at distance at most  $\epsilon$ . Therefore for all  $s < r$ :

$$\text{diam} \circ \text{Gauss}(W'_s) \leqslant \text{diam} \circ \text{Gauss}(W_{s+\epsilon}) + \epsilon.$$

Similarly, using (2.1),

$$\text{diam} \circ \text{Gauss}(W'_s) \geqslant \text{diam} \circ \text{Gauss}(W_{s-\epsilon}) - \epsilon.$$

As a consequence, if we write  $f = \vartheta(W)$  and  $g = \vartheta(W')$ ,

$$f(s - \epsilon) - \epsilon \leqslant g(s) \leqslant f(s + \epsilon) + \epsilon.$$

Hence  $\vartheta((D(r), \epsilon)^{\tilde{\Psi}}) \subset (\epsilon)^F$ .  $\square$

**Proposition 4.8.** *For each  $\delta > 0$ , the inclusion  $\tilde{\Psi}_d(B^n)^{\delta} \subset \tilde{\Psi}_d(B^n)$  has a homotopy inverse through a homotopy that fixes  $\tilde{\Psi}_d(B^n)^0$  pointwise.*

*Proof.* For a positive real number  $r$ , let  $f_{r,t} \in \text{Emb}(B^n, B^n)$  be the isotopy of embeddings  $f_{r,t}(x) = (t + (1-t)r)x$ , and let  $f_r = f_{r,0}$ . Let  $\sigma$  be given by Lemma 4.6. Define  $h: \tilde{\Psi}_d(B^n) \rightarrow \tilde{\Psi}_d(B^n)^{\delta}$  by  $W \mapsto f_{\sigma \circ \vartheta(W)}^{-1}(W)$ , which is continuous by Lemma 3.1, and is well-defined because

$$h(W) = \frac{1}{\sigma \circ \vartheta(W)}(W \cap B(\sigma \circ \vartheta(W)))$$



and

$$\begin{aligned} \text{diam} \circ \text{Gauss}\left(\frac{1}{\sigma \circ \vartheta(W)}(W \cap B(\sigma \circ \vartheta(W)))\right) &= \text{diam} \circ \text{Gauss}(W \cap B(\sigma \circ \vartheta(W))) \\ &= (\vartheta(W))(\sigma \circ \vartheta(W)) \leq \delta \end{aligned}$$

(the latter inequality follows from Lemma 4.6).

By Lemma 3.1, the map  $H_t: \tilde{\Psi}_d(B^n) \rightarrow \tilde{\Psi}_d(B^n)$  (resp.  $G_t: \tilde{\Psi}_d(B^n)^\delta \rightarrow \tilde{\Psi}_d(B^n)^\delta$ ) that sends a submanifold  $W$  to  $f_{\sigma \circ \vartheta(W), t}^{-1}(W)$  defines a homotopy between  $i \circ h$  and the identity (resp.  $h \circ i$  and the identity). It is clear that  $\tilde{\Psi}_d(B^n)^0$  remains fixed because the value of  $\varpi$  on an element  $W$  of  $\tilde{\Psi}_d(B^n)^0$  is the constant function with value 0, and so the value of  $\sigma$  on  $\varpi(W)$  is 1.  $\square$

## 5. THE SPACE OF LINEAR SUBMANIFOLDS

Let  $\nu: [0, 1] \rightarrow [0, 1]$  be a bump function that takes value 1 in a neighbourhood of 0 and value 0 on  $[\frac{1}{2}, 1]$  and whose slope is bounded by 7. If  $x \in B^n$ , we write  $\nu(x) := \nu(\|x\|)$ .

**Definition 5.1.** For each non-empty  $W \in \tilde{\Psi}_d(B^n)$  and each plane  $L \in \text{Gr}_d(\mathbb{R}^n)$ , define

$$\lambda(W, L) = \frac{1}{2 \int_W \nu(x) dx} \int_W \nu(x) d(L, T_x W)^2 dx. \quad (5.1)$$

As a consequence of Karcher's theorem [Kar77, Theorem 1.2], we have:

**Proposition 5.2.** *There exists a  $\delta > 0$  (that depends only on the metric of  $\text{Gr}_d(\mathbb{R}^n)$ ) such that if  $\text{diam} \circ \text{Gauss}(W) < \delta$ , then for all  $L \in \text{Gauss}(W)$ , the function  $\lambda(W, -)$  is convex in  $B_\delta(L)$ .*

Observe in addition that the global minima of  $\lambda(W, -)$  lie in the convex hule of  $\text{Gauss}(W)$ , which is contained in  $B_\delta(L)$ , and therefore  $\lambda(W, -)$  has a unique global minimum.

**Definition 5.3.** Let  $\delta$  be given by Proposition 5.2 theorem, and define the map

$$\mu: \tilde{\Psi}_d(B^n)^\delta \setminus \{\emptyset\} \longrightarrow \text{Gr}_d(\mathbb{R}^n)$$

whose value on a submanifold  $W$  is the minimum of  $\lambda(W, -)$  (cf. with the concept of *selection* [Mic51, p. 154]).

The main property of this function is that it assigns to each submanifold  $W$  whose Gauss map has small diameter, a plane  $\mu(W)$  that is near each tangent plane of  $W$  (so its orthogonal complement  $\mu(W)^\perp$  is transverse to  $W$ ):

$$d(\mu(W), T_x W) \leq \text{diam} \circ \text{Gauss}(W) \text{ for all } x \in W. \quad (5.2)$$

because otherwise, evaluation on (5.1) gives that

$$\lambda(W, T_x W) < \lambda(W, \mu(W)) \text{ for some } x \in W,$$

contradicting the definition of  $\mu(W)$ .

**Proposition 5.4.** *The map  $\mu$  is continuous.*

*Proof.* We will prove that the function  $\lambda(-, -)$  is continuous. Then, because  $\text{Gr}_d(\mathbb{R}^n)$  is a compact metric space, the adjoint  $\alpha: \tilde{\Psi}_d(B^n) \rightarrow \text{map}(\text{Gr}_d(\mathbb{R}^n), \mathbb{R})$  is continuous too. The target of this map is actually the subspace  $X$  of those maps that

- (1) have a unique global minimum,
- (2) are convex in the ball of radius  $\delta$  around the global minimum.

Taking the global minimum defines a continuous function  $\beta: X \rightarrow \text{Gr}_d(\mathbb{R}^n)$ , and therefore  $\mu = \beta \circ \alpha$  is continuous as well.

Let  $(W, L) \in \tilde{\Psi}_d(U) \times \text{Gr}_d(\mathbb{R}^n)$ , and let  $(K_t, \epsilon)^{\tilde{\Psi}}$  be a neighbourhood of  $W$ , where  $K$  is a disc of radius  $t$ . Let  $W' \in (K, \epsilon)^{\tilde{\Psi}}$  and let  $L' \in B_\eta(L)$ , the ball of radius  $\eta$  centered at  $L$ . Then, by Definition 2.2, there is a covering map  $q: Q \subset NW \rightarrow W$  over  $W \cap K$  such that

- (1)  $\exp_W(Q) \cap K \subset W' \cap K$ ,
- (2)  $\|f(x)\| + \|\tau \circ (Df)(x)\| < \epsilon$  for each local section  $f$  of the covering map.

Let  $t \geq \frac{3}{4}$  and let  $\epsilon < \frac{1}{4}$  and define  $V := W \cap K_t$  and  $V' := q^{-1}(W_t)$ . Then:

$$\begin{aligned} \int_W \nu(x) d(L, T_x W)^2 dx &= \int_V \nu(x) d(L, T_x W)^2 dx \\ \int_{W'} \nu(x) d(L, T_x W)^2 dx &= \int_{V'} \nu(x) d(L, T_x W)^2 dx \end{aligned}$$

because  $\nu(x) = 0$  if  $\|x\| > \frac{1}{2}$  and  $W \cap K_{\frac{1}{2}} \subset V \subset W$  and  $W' \cap K_{\frac{1}{2}} \subset V' \subset W'$ . Therefore we have that the difference  $|\lambda(W, L) - \lambda(W', L)|$  is

$$\begin{aligned} &= \left| \frac{1}{\int_W \nu(x) dx} \int_W \nu(x) d(L, T_x W)^2 dx - \frac{1}{\int_{W'} \nu(y) dy} \int_{W'} \nu(y) d(L', T_y W')^2 dy \right| \\ &= \left| \frac{1}{\int_V \nu(x) dx} \int_V \nu(x) d(L, T_x W)^2 dx - \frac{1}{\int_{V'} \nu(y) dy} \int_{V'} \nu(y) d(L', T_y W')^2 dy \right| \\ &= \left| \frac{1}{\int_V \nu(x) dx} \int_V \nu(x) d(L, T_x W)^2 dx - \frac{1}{\int_{V'} \nu(y) dy} \int_V \sum_{y \in q^{-1}(x)} \nu(y) d(L', T_y W')^2 \|\det J_y(q)\| dx \right|. \end{aligned}$$

Now, we write  $A = \int_V \nu(x) dx$ ,  $A' = \int_{V'} \nu(x) dx$ :

$$= \frac{1}{AA'} \left| \int_V A' \nu(x) d(L, T_x W)^2 dx - \int_V \sum_{y \in q^{-1}(x)} A \nu(y) d(L', T_y W')^2 \|\det J_y(q)\| dx \right|$$

Now, let  $k$  be the cardinality of  $q^{-1}(x)$ , and replace the second occurrence of  $A'$  by  $kA + |A' - kA|$ :

$$\leq \frac{1}{AA'} \left| \int_V (kA + |A' - kA|) \nu(x) d(L, T_x W)^2 dx - \int_V \sum_{y \in q^{-1}(x)} A \nu(y) d(L', T_y W')^2 \|\det J_y(q)\| dx \right|$$

Now, move the term  $kA \nu(x) d(L, T_x W)^2$  to the second integral:

$$\leq \frac{1}{AA'} \left| \int_V |A' - kA| \nu(x) d(L, T_x W)^2 dx - \int_V \sum_{y \in q^{-1}(x)} |A \nu(x) d(L, T_x W)^2 - A \nu(y) d(L', T_y W')^2| \|\det J_y(q)\| dx \right|$$

rearranging the  $A$  and the  $A'$ :

$$= \frac{|A' - kA|}{A'} \int_V \nu(x) d(L, T_x W)^2 dx - \frac{1}{A'} \int_V \sum_{y \in q^{-1}(x)} |\nu(x) d(L, T_x W)^2 - \nu(y) d(L', T_y W')^2| \|\det J_y(q)\| dx$$

and using that  $ab - cd = a(b - d) + (a - c)d$  with

- (1)  $a = \nu(x)$ ,
- (2)  $b = d(L, T_x W)^2$
- (3)  $c = \nu(y) \|\det J_y(q)\|$
- (4)  $d = d(L', T_y W')^2$

we have that the last display is

$$\begin{aligned} & \frac{|A' - kA|}{A'} \int_V \nu(x) d(L, T_x W)^2 dx \\ & - \frac{1}{A'} \int_V \sum_{y \in q^{-1}(x)} |\nu(x)| d(L, T_x W)^2 - d(L', T_y W')| + |\nu(x) - \nu(y)| |\det(J_y(q))| |d(L', T_y W')^2| dx \end{aligned} \quad (5.3)$$

First, using the triangle inequality one has that

$$|d(L, T_x W) - d(L', T_y W')| \leq d(L, L') + d(T_x W, T_y W'),$$

and both If we write this inequality as  $|a - b| \leq c + d$ , multiplying both sides by then we have that

$$|a^2 - b^2| \leq (a + b)(c + d)$$

and here  $c < \eta$ ,  $d < \tan(\epsilon) < \epsilon$  (see Remark 2.4 (1)) and  $a, b \leq \pi/2$ . In addition,  $\nu(x) \leq 1$  therefore

$$\nu(x) |d(L, T_x W) - d(L', T_y W')| \leq \pi \cdot (\epsilon + \eta). \quad (5.4)$$

Second, using that the slope of  $\nu$  is at most 7, that  $\|x\| - \|y\| < \epsilon$  and that  $\nu(x) \leq 1$ , we have that

$$\begin{aligned} |\nu(x) - \nu(y) \det(J_y(q))| & \leq |\nu(x) - \nu(y)| + |\nu(y)(1 - \det J_y(q))| \\ & \leq 7\epsilon + |1 - \det J_y(q)| \end{aligned}$$

As a consequence (and using that  $d(L, T_x W)^2 \leq \pi/2$ ), (5.3) is bounded by

$$\frac{|A' - kA|}{A'} A + k\pi \frac{((\epsilon + \eta) + (7\epsilon + |1 - \det J_y(q)|)/2)}{A'} \text{vol}(V). \quad (5.5)$$

Finally,  $|A' - kA|$  is bounded by

$$\int_{V'} \nu(x) dx - k \int_V \nu(x) dx = \int_V \sum_{y \in q^{-1}(x)} \nu(y) dx - k \int_V \nu(x) dx = \int_V \sum_{y \in q^{-1}(x)} (\nu(x) - \nu(y)) dx.$$

Since  $\nu$  has slope at most 7 and  $d(x, y) < \epsilon$ , we have that the latter expression is bounded by  $7k\epsilon \text{vol}(V)$ . Using this, it follows that  $A' \geq |kA - 7k\epsilon \text{vol}(V)| \sim kA$  if  $\epsilon$  is very small. Hence (5.5) is bounded by:

$$\frac{7k\epsilon \text{vol}(V)}{|kA - 7k\epsilon \text{vol}(V)|} A + k\pi \frac{((\epsilon + \eta) + (7\epsilon + |1 - \det J_y(q)|)/2)}{A'} \text{vol}(V)$$

which can be made arbitrarily small taking  $\epsilon$  and  $\eta$  small enough (notice that  $|1 - \det J_y(q)|$  is bounded by a continuous function of  $\epsilon$  which does not depend on  $W$ ).  $\square$

**Proposition 5.5.** *Let  $\delta$  be given by Proposition 5.2, and assume that  $\delta < \pi/2$ . Then the inclusion  $\tilde{\mathcal{L}}_d(B^n) \subset \tilde{\Psi}_d(B^n)^\delta$  is a deformation retract.*

*Proof.* If  $P \in \text{Gr}_d(\mathbb{R}^n)$ , and  $\pi, \pi^\perp$  are the projections onto  $P$  and its orthogonal complement, we let  $g_{t,P}(x) = t \cdot \pi(x) + \pi^\perp(x)$  for all  $x \in B^n$ . This defines a continuous map

$$g: [0, 1] \times \text{Gr}_d(\mathbb{R}^n) \times B^n \longrightarrow B^n.$$

Let  $G_t: \tilde{\Psi}_d(B^n)^\delta \rightarrow \tilde{\Psi}_d(B^n)$  be the homotopy

$$G_t(W) = \begin{cases} g_{t, \mu(W)}^{-1}(W) & \text{if } t > 0 \\ \bigcup_{x \in \mu(W)^\perp \cap W} (x + \mu(W)) \cap B^n & \text{if } t = 0 \\ \emptyset & \text{if } W = \emptyset, \end{cases}$$

where  $\mu(W)^\perp \cap W$  is discrete because  $W$  intersects  $\mu(W)^\perp$  transversely: By (5.2), we have that for all  $x \in W$ ,

$$d(T_x W, \mu(W)) \leq \text{diam} \circ \text{Gauss}(W) < \delta < \pi/2. \quad (5.6)$$

By Lemma 3.1 and because  $g$  and  $\mu$  are continuous,  $G_t$  is continuous for  $t > 0$ . Let us see now what happens when  $t = 0$ .

We identify  $NG_0(W) = G_0(W) \times \mu(W)^\perp \subset \mathbb{R}^n$ , so that the exponential map  $\exp_{G_0}^{-1}: B^n \rightarrow \mathbb{R}^n$  is the inclusion. Then by (5.6), we have that the restriction  $q_t$  of the projection of  $NG_0(W)$  onto  $G_0(W)$  to  $G_t(W)$  is a covering map and that if  $f_t$  is a local section of  $q_t$ ,

$$\begin{aligned} \|f_t(x)\| &= f_1(tx) = d_0(tx, G_0(W)) \\ \|\tau \circ (Df_t)(x)\| &= t \cdot \|\tau \circ (Df_1)(tx)\| \\ &= t \cdot \tan(d_1(\mu(W), T_{tx}(W))) \leq t \cdot \tan(\delta). \end{aligned} \quad (5.7)$$

For each  $\epsilon > 0$ , let  $r_\epsilon$  be so small that if  $x \in B_{r_\epsilon}(\mu(W)^\perp) \cap W$ , then  $d_0(x, G_0(W)) < \epsilon/2$ . Then by the formula above, if  $t < r_\epsilon$ , then  $\|f_t(x)\| < \epsilon/2$  for all  $x$ . If in addition  $t < \frac{\epsilon}{2 \tan(\delta)}$ , then we have that  $\|\tau \circ (Df_t)(x)\| < \epsilon/2$ . Therefore, if  $(K, \epsilon)$  is a neighbourhood of  $G_0(W)$ ,

$$t < \lambda_\epsilon := \min\{r_\epsilon, \frac{\epsilon}{2 \tan(\delta)}\} \Rightarrow G_t(W) \in (K, \epsilon).$$

If  $(K, \epsilon)^{\tilde{\Psi}}$  is a basic neighbourhood of  $W$ , we will denote it by  $(K, \epsilon)_W$  during the rest of the proof. Observe that if  $\epsilon' > 0$  and  $K$  is compact, then

$$G_t((K', \epsilon')_W) \subset (K', \epsilon')_{G_t(W)},$$

$$(K', \epsilon')_{G_t(W)} \subset (K, \epsilon)_{G_0(W)} \Rightarrow (K', \epsilon')_{G_s(W)} \subset (K, \epsilon)_{G_0(W)} \text{ for all } s \leq t.$$

Let  $(K, \epsilon)_{G_0(W)}$  be a neighbourhood of  $G_0(W)$ . We have seen that  $G_t(W) \in (K, \epsilon)_{G_0(W)}$  for all  $t \leq \lambda_\epsilon$ , so we may pick a neighbourhood  $(K', \epsilon')_{G_{\lambda_\epsilon}(W)}$  of  $G_{\lambda_\epsilon}(W)$  contained in  $(K, \epsilon)_{G_0(W)}$ . Then, for all  $t < \lambda_\epsilon$ , we have that  $(K', \epsilon')_{G_t(W)}$  is a neighbourhood of  $G_t(W)$  that is contained in  $(K, \epsilon)_{G_0(W)}$ .

Then we have that  $G(t, W) = G_t(W)$  sends the neighbourhood  $[0, \lambda_\epsilon/2) \times (K', \epsilon')_W$  of  $(0, W)$  into the neighbourhood  $(K, \epsilon)_{G_0(W)}$  of  $G_0(W)$ . Hence  $G_t$  is continuous around  $t = 0$ .

In particular,  $g := G_0$  is well-defined and lands in  $\tilde{\mathcal{L}}_d(B^n)$ , and  $i \circ g$  is homotopic to the identity through the homotopy  $G_t$ . On the other hand,  $g \circ i$  is the identity because the value of  $\mu$  on an affine plane  $P$  is obtained by translating  $P$  to the origin, and so  $G_t$  applied to a plane is constant in  $t$ , hence  $g$  restricts to the identity on  $\tilde{\mathcal{L}}_d(B^n)$ .  $\square$

## 6. THE HOMOTOPY TYPE OF THE SPACE OF LINEAR SUBMANIFOLDS

**Proposition 6.1.** (cf. [Gal11, Lemma 6.1]) *The space  $\tilde{\mathcal{L}}_d(B^n)$  is weakly contractible.*

*Proof.* Let  $C_d(\mathbb{R}^n) \subset \tilde{\mathcal{L}}_d(\mathbb{R}^n)$  be the subspace of those non-empty unions of affine planes, all of whose origins (i.e. their closest points to the origin of  $\mathbb{R}^n$ ) are at distance at most 1 from the origin of  $\mathbb{R}^n$ . This is a closed subset. Let  $U \subset \tilde{\mathcal{L}}_d(\mathbb{R}^n)$  be the subspace of those (possibly empty) unions of planes that do not contain the origin. Then, there is a pushout square

$$\begin{array}{ccc} U \cap C_d(\mathbb{R}^n) & \xrightarrow{\quad} & U \\ \downarrow & & \downarrow \\ C_d(\mathbb{R}^n) & \xrightarrow{\quad} & \tilde{\mathcal{L}}_d(\mathbb{R}^n) \end{array}$$

which is also a homotopy pushout square because the upper horizontal arrow is a cofibration. Now, a point in  $C_d(\mathbb{R}^n)$  is a collection of parallel planes, and remembering the underlying linear plane defines a map  $C_d(\mathbb{R}^n) \rightarrow \text{Gr}_d(\mathbb{R}^n)$  that is also a fibre bundle. Its fibre over a plane  $P$  is the space  $C_0(P^\perp)$ , which is the Ran space of  $P^\perp$  and is well-known to be weakly contractible [Gai13, Appendix]. Therefore  $C_d(\mathbb{R}^n) \simeq \text{Gr}_d(\mathbb{R}^n)$ . The same argument proves that  $U \cap C_d(\mathbb{R}^n) \simeq \text{Gr}_d(\mathbb{R}^n)$ , and since the left vertical map is a map over  $\text{Gr}_d(\mathbb{R}^n)$ , it is a weak homotopy equivalence. On the other hand,  $U$  is contractible as the homotopy  $(W, t) \mapsto \frac{1}{1-t}W$  defines a contraction of  $U$  to the empty submanifold. As a consequence, the homotopy pushout  $\tilde{\mathcal{L}}_d(\mathbb{R}^n)$  is weakly contractible as well.  $\square$

**Proposition 6.2.** *The subspace  $\text{Th}(\gamma_d^\perp(B^n))$  of  $\mathcal{L}_d(B^n)$  that consists of connected or empty submanifolds, is a strong deformation retract.*

*Proof.* If  $W \in \mathcal{L}_d(B^n)$ , let  $W_{\text{first}}$  be the closest component of  $W$  to the origin of  $B^n$ , and let  $f(W) > 0$  be the distance from  $W \setminus W_{\text{first}}$  to the origin. This is a continuous function  $f: \mathcal{L}_d(B^n) \rightarrow [0, 1]$ . Then  $h(W) = \frac{1}{f(W)}W$  defines a map  $h: \mathcal{L}_d(B^n) \rightarrow \text{Th}(\gamma_d^\perp(B^n))$  and  $h \circ i$  is the identity and  $i \circ h$  is homotopic to the identity through the homotopy  $H_t(W) = \frac{1}{(1-t)+tf(W)}W$ .  $\square$

## 7. MICROFLEXIBILITY

The spaces  $\Psi_d(U)$  and  $\tilde{\Psi}_d(U)$  glue together to form sheaves  $\Psi_d(-)$  and  $\tilde{\Psi}_d(-)$  on  $\mathbb{R}^n$ : If  $U \subset U'$  is a pair of open subsets then the restriction map  $\tilde{\Psi}_d(U') \rightarrow \tilde{\Psi}_d(U)$  sends a submanifold  $W$  to the intersection  $W \subset W'$ , and we have proven that these are continuous in families in Lemmas 3.1 and 3.2. A sheaf of topological spaces in  $\mathbb{R}^n$  extends canonically to a sheaf of topological spaces in the site of manifolds and open embeddings [RW11, Theorem 3.3].

At this point, one is tempted to use the methods in the latter article to extend our theorem to the space of merging submanifolds in an arbitrary open manifold: In that article, it was proven that the sheaf  $\Psi(-)$  on a manifold  $M$  is  $\text{Diff}(M)$ -equivariant and that it is *microflexible*. By a theorem of Gromov [Gro86], this automatically implies that for connected non-compact manifolds  $M$  a certain map

$$\Psi(M) \longrightarrow \Gamma(\Psi^{\text{fib}}(TM) \rightarrow M)$$

is a homotopy equivalence. The space on the right is the space of sections of the fibrewise space of submanifolds of the tangent bundle of  $M$ . By the Galatius–Randal-Williams theorem, the fibre over each point is homotopy equivalent to the Thom space  $\text{Th}\gamma_{d,n}^\perp$ .

The sheaves  $\tilde{\Psi}_d(-)$  and  $\tilde{\Psi}_d(-; X)$  are in fact  $\text{Diff}(M)$ -equivariant, and if they were also microflexible, then one could deduce that certain maps

$$\begin{aligned} \tilde{\Psi}_d(M) &\longrightarrow \Gamma(\tilde{\Psi}_d^{\text{fib}}(TM) \rightarrow M) \\ \tilde{\Psi}_d(M) &\longrightarrow \Gamma(\tilde{\Psi}_d^{\text{fib}}(TM; X) \rightarrow M) \end{aligned}$$

are homotopy equivalences. But this is a castle in the sky:

**Proposition 7.1.** *The sheaves  $\tilde{\Psi}_d(-)$  and  $\tilde{\Psi}_d(-; X)$  are not microflexible.*

Let us recall first the definition of microflexibility.

**Definition 7.2.** A sheaf  $\Phi$  on a manifold  $M$  is microflexible if for each pair  $C' \subset C$  of compact subspaces of  $M$ , and each pair  $C' \subset U', C \subset U$  of open subsets of  $M$

such that  $U' \subset U$ , and for each diagram

$$\begin{array}{ccc} P \times \{0\} & \xrightarrow{f} & \Phi(U) \\ \downarrow & & \downarrow r \\ P \times [0, 1] & \xrightarrow{h} & \Phi(U') \end{array}$$

there exists an  $\epsilon > 0$  and a pair of open subsets  $C' \subset V' \subset U'$  and  $C \subset V \subset U$  such that  $V' \subset V$ , and a dashed arrow

$$\begin{array}{ccccc} P \times \{0\} & \longrightarrow & \Phi(U) & \longrightarrow & \Phi(V) \\ \downarrow & & & \nearrow \text{dashed} & \downarrow \\ P \times [0, \epsilon) & \longrightarrow & \Phi(U') & \longrightarrow & \Phi(V') \end{array}$$

*Proof.* As for the main theorem, we only give the proof for the sheaf  $\tilde{\Psi}_d(-)$ , the proof for  $\tilde{\Psi}_d(-; X)$  being exactly the same.

Let  $W'$  be a connected compact submanifold of  $\mathbb{R}^n$ , and let  $C' = U'$  be a tubular neighbourhood of  $W'$  (which we implicitly identify with  $NW$  from now on).

Let  $W'' \subset W \subset W'$  be codimension 0 submanifolds such that  $W''$  is closed as a subset and  $W$  is open, and the first inclusion is a homotopy equivalences.

Let  $U$  and  $C$  be the restrictions of  $U'$  to  $W$  and  $W''$  respectively.

Let  $C_k(NW)$  be the fibrewise configuration space of  $k$  unordered points in the normal bundle of  $W$ . A section  $f$  of this bundle defines

- (1) a  $k$ -sheeted covering of  $W$  and
- (2) an element in  $\tilde{\Psi}_d(NW) \subset \tilde{\Psi}_d(U)$ .

Since the fibre of  $NW$  is a vector space, we can multiply any subset of it by a real number. Then we can define a path

$$[0, 1] \longrightarrow \tilde{\Psi}_d(NW)$$

by sending  $t > 0$  to  $t \cdot f(W)$  and  $t = 0$  to  $W$ . If we have a family of sections of  $C_k(NW)$  indexed by  $P$ , we obtain a map

$$g: P \times [0, 1] \longrightarrow \tilde{\Psi}_d(NW)$$

whose restriction to  $P \times \{0\}$  is constant. Suppose now that we have a microflexible solution for the diagram

$$\begin{array}{ccc} P \times \{0\} & \xrightarrow{c} & \tilde{\Psi}_d(U') \\ \downarrow & & \downarrow r \\ P \times [0, 1] & \xrightarrow{g} & \tilde{\Psi}_d(U) \end{array}$$

where  $c$  is the constant map with value  $W'$ . This means that we can find an  $\epsilon > 0$  and an open subset  $C \subset V \subset U$  such that

$$\begin{array}{ccccc} P \times \{0\} & \xrightarrow{c} & \tilde{\Psi}_d(U) & \longrightarrow & \tilde{\Psi}_d(V) \\ \downarrow & & & \nearrow h & \downarrow \\ P \times [0, \epsilon) & \xrightarrow{g} & \tilde{\Psi}_d(U') & \longrightarrow & \tilde{\Psi}_d(U') \end{array}$$

Then, for small values of  $\delta \in [0, \epsilon)$ , the map  $h$  takes values in the space of sections of  $C_k(NW)$  (the  $k$  is determined because  $h$  is extending the section  $g$  that takes values in  $C_k(NW)$ ), and because the inclusion  $W \subset W'$  induces an epimorphism in components), and therefore defines for each  $p \in P$  a  $k$ -sheeted covering of  $W$ . As

a consequence, the above solution gives also a lift to the following diagram (where  $\text{Cov}(W)$  denotes the space of finite sheeted coverings of  $W$ ):

$$\begin{array}{ccc} & & \text{Cov}(W') \\ & \nearrow h & \downarrow \\ P & \xrightarrow{g} & \text{Cov}(W). \end{array}$$

But this would mean that any family of coverings of  $W$  can be extended to a family of coverings of  $W'$  which is false (for instance, if  $W' = S^2$  and  $W$  is a equatorial band in  $S^2$ ).  $\square$

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